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THE BOUNDARY-LAYER REGIME FOR CONVECTION IN A VERTICAL POROUS LAYER

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Abstract—Buoyancy-driven convection in a differentially heated vertical porous layer is studied theoretically by the method developed by Gill [5]. The model is of finite extent, and the temperature difference between the vertical walls is assumed to be large. Satisfactory agreement with experiment has been obtained for the interior temperature distribution and the Nusselt number. The applied method is also extended to include some effects of a variable viscosity. This is shown to introduce asymmetry into the solutions.

NOMENCLATURE

- L, width of model;
- H, height of model;
- d, characteristic grain diameter;
- k, permeability of porous medium;
- g, acceleration of gravity;
- x_*, y_*, z_*, Cartesian coordinates;
- $\mathbf{v}_*$, $(=u_*, w_*)$, two-dimensional velocity vector;
- T_*, temperature;
- ΔT, dimensional temperature difference between the vertical walls;
- $f^{\pm}$, $(=v^{\pm}/v_r)$, defined by (2.8);
- T_0, dimensionless temperature in the core;
- s, q, defined by (3.7);
- Re, Reynolds number;
- Pr, Prandtl number, $= \nu_r/\kappa_m$;
- Pe, Peclet number, $= PrRe$;
- Ra, Rayleigh number, $= kg\gamma\Delta TL/\kappa_m \nu_r$;
- Nu, Nusselt number.

Greek symbols

- β, dimensionless temperature gradient;
- γ, coefficient of volume expansion;
- δ, defined by (2.5);
- η, θ, defined by (2.7);
- κ_m, thermal diffusivity;
- λ, defined by (3.3);
- ν, kinematic viscosity;
- ν_r, reference viscosity;
- ν_1, ν_2, viscosities at the hot and cold wall, respectively;
- ξ, defined by (3.6);
- ψ_*, stream function;
- ψ_0, dimensionless stream function in the core.

Subscripts

- $*$, dimensional quantities;
- m, solid-fluid mixture;
- A, average values.

Superscripts

- $'$, derivation with respect to z_*/H;
- $^+, ^-$, denotes left- and right-hand boundary layer, respectively.

1. INTRODUCTION

It is well known that an appreciable insulating effect may be achieved by placing a porous material (fibre glass, say) in the unventilated gap between vertical walls. This is due to the fact that multi-cellular convection does not occur in this case, as shown theoretically by Gill [1] for a porous slab of infinite height. Also for a slab of finite height, observations indicate a unicellular motion (Klarsfeld [2], Bories and Combarnous [3]), and so does the analysis by Chan *et al.* [4]. Observations further show that when the temperature difference between the walls, or equivalently, the Rayleigh number, is sufficiently increased, the basic flow exhibits boundary-layer character. Distinct thermal boundary layers develop along the vertical walls, while the core-region is characterized by a positive vertical temperature gradient.

In the present paper this boundary-layer flow is studied. The applied method is similar to that developed by Gill [5] for the analogous fluid problem. In the present study the method is extended to include some effects of a variable kinematic viscosity. This is motivated by the fact that ν in practise may vary considerably due to the large values of ΔT often involved in this type of flow.

2. GOVERNING EQUATIONS

Consider natural convection in an enclosed porous medium with rectangular, impermeable boundaries. L and H are the width and the height, respectively, of the model (Fig. 1). The depth, in the y_*-direction, is infinite. The vertical walls are taken to be perfect heat conductors and maintained at the temperatures $\Delta T/2$ and $-\Delta T/2$, respectively, while the horizontal end-walls are insulating.

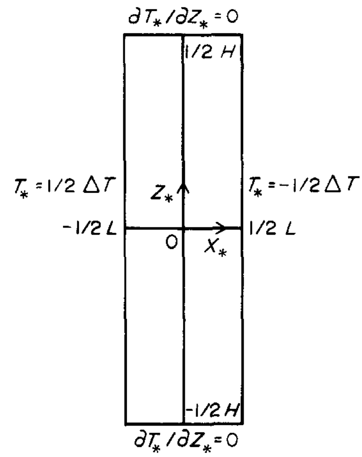

FIG. 1. The porous model.

Making the Boussinesq approximation, the equations of vorticity, heat and continuity for stationary two-dimensional motion can be stated as follows, respectively

$$\frac{\partial}{\partial x_*}(\nu w_*) - \frac{\partial}{\partial z_*}(\nu u_*) = kg\gamma\frac{\partial T_*}{\partial x_*} \tag{2.1}$$

$$u_*\frac{\partial T_*}{\partial x_*} + w_*\frac{\partial T_*}{\partial z_*} = \kappa_m\left(\frac{\partial^2 T_*}{\partial x_*^2} + \frac{\partial^2 T_*}{\partial z_*^2}\right) \tag{2.2}$$

$$\frac{\partial u_*}{\partial x_*} + \frac{\partial w_*}{\partial z_*} = 0 \tag{2.3}$$

where we have allowed for a variable viscosity in (2.1). This is relevant, since in practice the viscosity may vary rapidly with temperature.

Introducing the stream function ψ_* by $u_* = -\partial\psi_*/\partial z_*$ and $w_* = \partial\psi_*/\partial x_*$, a balance in (2.2) between convection and conduction in the boundary layers requires that

$$\psi_* \sim \kappa_m H/\delta \tag{2.4}$$

where δ is the thickness of the boundary layers at the vertical walls. From (2.1) a balance between buoyancy and vorticity yields

$$\delta^2 \sim H\nu_r\kappa_m/kg\gamma\Delta T \tag{2.5}$$

where ν_r is a reference viscosity.

Boundary-layer variables are defined by taking $\delta = (H\nu_r\kappa_m/kg\gamma\Delta T)^{1/2}$, H, ΔT, $\kappa_m H/\delta$ as scales of horizontal length, vertical length, temperature and stream function, respectively. These scales are also assumed to be characteristic for the motion in the core, except that the horizontal length is taken to be the width L. Requiring that $\delta \ll L$ and $\delta \ll H^2/L$, it follows analogously to Gill [5] that, as a first approximation, the temperature and the stream function in the core can be written

$$\begin{aligned} T &= T_0(z) \\ \psi &= \psi_0(z). \end{aligned} \tag{2.6}$$

Defining boundary-layer variables by

$$\begin{aligned} T &= T_0(z) + \theta(x, z) \\ \psi &= \psi_0(z) + \eta(x, z) \end{aligned} \tag{2.7}$$

where $\theta, \eta \to 0$ as $x \to \infty$, the approximate forms of (2.1)–(2.3) valid in the boundary layers may be written

$$f^{\pm}w = \theta \tag{2.8}$$

$$\pm u\theta_x + wT_z = \theta_{xx} \tag{2.9}$$

$$u = -\psi_0' - \eta_z, \quad w = \pm\eta_x \tag{2.10}$$

where $f^{\pm} = \nu^{\pm}/\nu_r$, and the plus and minus signs correspond to the left- and right-hand boundary layers, respectively.

3. METHOD OF SOLUTION

The nonlinear system (2.8)–(2.10) will be solved by the modified Oseen technique developed by Gill [5]. Essentially, this means that u and T_z in (2.9) are replaced by average values $u_A(z)$ and $T_A'(z)$. When the viscosity is constant, i.e. $f = 1$, the solutions exhibit centro-symmetrical properties, and hence only one boundary layer needs to be considered. In this case the analysis is quite similar to Gill's for infinite Prandtl number. It is in fact simpler, since $-w_{xx}$ in the vorticity equation now, due to Darcy's law, is replaced by w. Also the results show qualitatively the same behavior, see Figs. 2–5 (solid lines).

When the viscosity variation is taken into account, the results do not follow so readily. We therefore give some main steps in the analysis. In this case there is no centro-symmetry, and both boundary layers must be considered. An Oseen approximation of the vorticity equation is achieved by replacing $f = \nu/\nu_r$ by an average value $f_A(z)$. Substituting the average values u_A, T_A' and f_A into (2.8) and (2.9), we finally obtain

$$\theta = T - T_0 = (\pm\tfrac{1}{2} - T_0)e^{-\lambda x} \tag{3.1}$$

$$w = \frac{(\pm\frac{1}{2} - T_0)e^{-\lambda x}}{f_A^{\pm}} \tag{3.2}$$

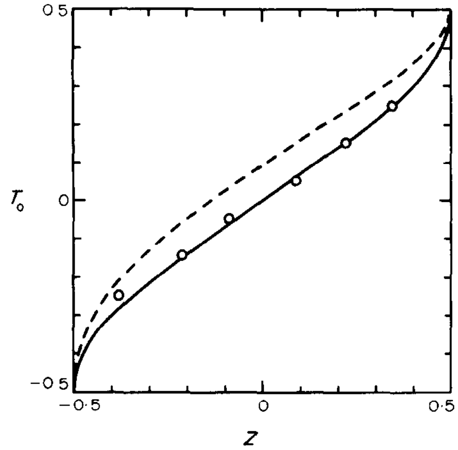

FIG. 2. The temperature $T_0(z)$ in the core. The solid line corresponds to constant viscosity, $\xi = 0$, while the broken line is for $\xi = 0{\cdot}25$, i.e. $\nu_2 = 3\nu_1$. The experimental points are from Klarsfeld [2] (Fig. 11(E10) where $Ra = 1298$ and $H/L = 2{\cdot}25$). Here the variation of ν is not significant (less than 10 per cent).

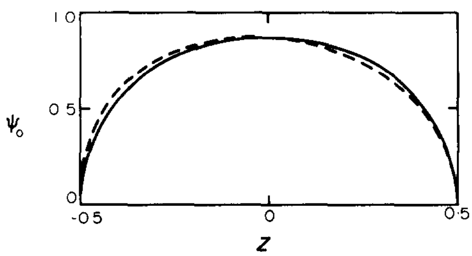

FIG. 3. The stream function $\psi_0(z)$ for the core. Solid and broken lines correspond to $\xi = 0$ and $\xi = 0{\cdot}25$, respectively.

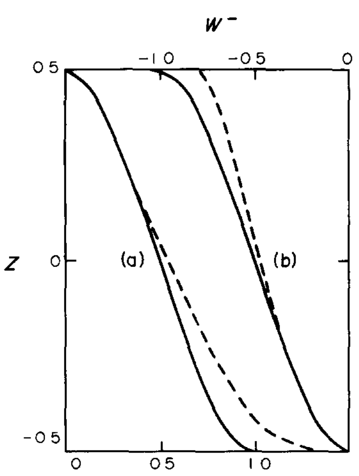

FIG. 4. Vertical velocities at (a) the left-hand wall, w^+, and (b) the right-hand wall, w^-. Solid and broken lines correspond to $\xi = 0$ and $\xi = 0{\cdot}25$, respectively.

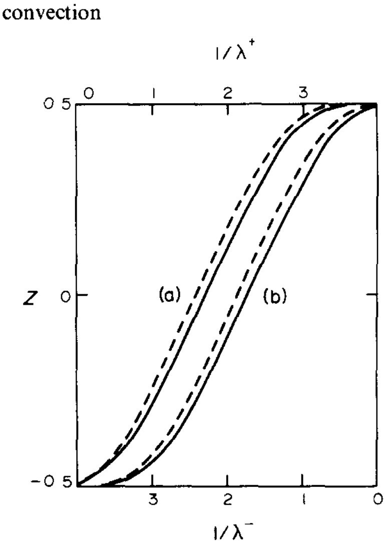

FIG. 5. Boundary-layer thicknesses at (a) the left-hand wall, $1/\lambda^+$, and (b) the right-hand wall, $1/\lambda^-$. Solid and broken lines correspond to $\xi = 0$ and $\xi = 0{\cdot}25$, respectively.

where

$$\lambda = -\tfrac{1}{2}(\pm u_A)+\tfrac{1}{2}\sqrt{(u_A^2+4T_A'/f_A^\pm)} \tag{3.3}$$

Here u_A, T_A' are related to the core values $-\psi_0'$, T_0' through conditions obtained by integrating the continuity and heat equations (2.10) and (2.9) across the boundary layers. By aid of (3.1) and (3.2), this gives

$$\pm\psi_0 = \frac{\pm\frac{1}{2}-T_0}{\lambda f_A^\pm} \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(\lambda f_A^\pm) = -\frac{2\lambda^3(f_A^\pm)^2}{\pm\frac{1}{2}-T_0} \tag{3.5}$$

where a plus–minus superscript also should be understood in λ.

To solve (3.4) and (3.5) we must assume a relation between the temperature and the viscosity. When the temperature varies considerably, it is not possible to derive any simple general relationship between T and ν. To retain some general effects of a variable viscosity, and still have a tractable mathematical problem, we have to make some simplifying assumptions. Accordingly we take the viscosity to vary linearly over the boundary layers, and to be independent of height. If ν_1 and ν_2 are the viscosities at the hot and cold wall, respectively, and $\nu_r = (\nu_1+\nu_2)/2$ is the mean viscosity, we obtain

$$\begin{aligned} f_A^+ &= \nu_A^+/\nu_r = 1-\xi \\ f_A^- &= \nu_A^-/\nu_r = 1+\xi \end{aligned} \tag{3.6}$$

where

$$\xi = \frac{1}{2}\frac{\nu_2 - \nu_1}{\nu_2 + \nu_1}.$$

Introducing (3.6) into (3.3), and defining new variables by

$$s = \lambda^+ + \lambda^- \qquad q = (\lambda^- - \lambda^+)/s, \tag{3.7}$$

(3.4) and (3.5) yield four relations involving T_0, ψ_0, s and q. These finally reduce to

$$T_0 = \frac{1}{2}\frac{q+\xi}{1+\xi q} \tag{3.8}$$

$$\psi_0 = C_1 \frac{1-q^2}{1+\xi q} \tag{3.9}$$

and

$$\frac{z}{C_1^2} = \frac{1}{\xi}\left[\frac{q}{\xi} - \frac{1}{2}q^2 - \left(\frac{1-\xi^2}{\xi^2}\right)\ln(1+\xi q)\right] + C_2 \tag{3.10}$$

where C_1 and C_2 are constants of integration. Actually these should have been determined by matching with the solutions valid in the horizontal boundary layers. However, effects due to their presence are neglected in this analysis. Analogous to Gill [5], we then take the inner solution to be valid at the horizontal end-walls, i.e. $\psi_0(\pm\frac{1}{2}) = 0$. From (3.9) this implies that $q(z = \pm\frac{1}{2}) = \pm 1$, which enables us to determine C_1 and C_2 from (3.10).

Finally we mention that the solutions have singularities in the corners $(-\frac{1}{2}L, -\frac{1}{2}H)$ and $(\frac{1}{2}L, \frac{1}{2}H)$. This is analogous to the result in [5].

4. RESULTS AND DISCUSSION

In Figs. 2–5 T_0, ψ_0 etc. are shown as functions of z. For constant viscosity, $\xi = 0$ (solid lines), the solutions exhibit centro-symmetrical properties, which is in accordance with the system of equations and boundary conditions for that particular case. The broken lines in Figs. 2–5 correspond to $\xi = 0.25$, which means that the viscosity increases by a factor 3 from the hot to the cold wall. The figures clearly show that a variable ν introduces asymmetry into the solutions, as suggested by Gill.

From Fig. 2 we observe that the interior temperature distribution is close to a straight line in the central part of the layer. The plotted points are experimental values taken from Klarsfeld [2] using chlorobenzéne as saturating fluid. The variation of ν is not significant in his experiment, being less than 10 per cent over the layer. It is seen that the theoretical curve corresponding to constant viscosity agrees well with the experiments. For the temperature gradient β in the middle of the layer, our analysis for constant viscosity gives the value 0·67. The experiment plotted in Fig. 2 gives approximately $\beta = 0{\cdot}69$. For comparison we mention that Hart [6] has measured a gradient of 0·62 for the similar problem in a fluid layer.

The value of the stream function ψ_0 in the core is plotted in Fig. 3. Introducing the horizontal core velocity $u_0 = -\psi_0'$, we see that most of the mass flux across the core takes place near the upper and lower boundaries. This is in accordance with the observations in [2]. At the horizontal boundaries u_0 tends to infinity, and so does also the temperature gradient (Fig. 2). This occurs, as pointed out by Gill, because the effect of boundary layers on the horizontal end-walls have been neglected. In practice, then, diffusion will limit the velocity and the temperature gradient in these regions.

In Fig. 4 the vertical velocities w^+ and w^- at the left- and right-hand walls respectively, are plotted as functions of z. In Fig. 5 a similar plot is done for the boundary-layer thicknesses $1/\lambda^+$ and $1/\lambda^-$. We observe that a variable viscosity results in a higher velocity and a thinner boundary layer at the hot wall, and vice versa at the cold wall. This conforms to the observations by Elder [7] in a fluid slot.

The heat transfer across the layer may be expressed by the Nusselt number

$$Nu = \frac{1}{\Delta TH/L}\int_{-\frac{1}{2}H}^{+\frac{1}{2}H} -\left(\frac{\partial T_*}{\partial x_*}\right)_{x_* = -\frac{1}{2}L} dz_* \tag{4.1}$$

which is the ratio of the total heat transport to the heat transferred by pure conduction. For constant viscosity, the integration yields

$$Nu = \frac{\sqrt{3}}{3}\left(\frac{L}{H}\right)^{1/2} Ra^{1/2}. \tag{4.2}$$

Bories and Combarnous [3] report an empirical formula for the Nusselt number in a vertical layer involving L/H (in our notation) and Ra to the powers of 0·397 and 0·625, respectively, including all their experiments. However, only for their last run ($Ra = 520$) the flow exhibited boundary-layer character. Accordingly the proposed formula cannot be valid in the limit of large Ra. It seems plausible then, as the Rayleigh number increases, that both exponents should tend to 0·5 as a limit.

It may also be worth mentioning that by assuming a constant vertical temperature gradient β in the whole layer, as in [6] or [7], we easily derive that

$$Nu = \tfrac{1}{2}\beta^{1/2}Ra^{1/2} \tag{4.3}$$

when βRa is large. Taking now $\beta \propto L/H$ in the asymptotic case, we arrive at a formula similar to (4.2).

Finally, we emphasize that the assumption (3.6) for the viscosity is rather crude. When relevant experimental data becomes available, such that real comparisons can be made, a more realistic viscosity variation should possibly be applied in (3.4)–(3.5).

Before closing, it should be noted that several conditions must be satisfied to assure the validity of the present analysis. Firstly, the boundary-layer thickness must be much smaller than the horizontal and vertical extent of the model, which leads to

$$\left(\frac{H}{L}\right)^{1/2} \ll Ra^{1/2}. \tag{4.4}$$

Secondly, for Darcy's law to be valid and, at the same time, thermal dispersion effects to be neglected, the (grain) Reynolds and Peclet numbers should not exceed unity in the boundary layers. Thirdly, the characteristic grain diameter must be smaller than the boundary-layer thickness. This implies, respectively, that

$$\frac{d}{L} < 2PrRa^{-1}, \quad 2Ra^{-1}, \quad \left(\frac{H}{L}\right)^{1/2} Ra^{-1/2}. \tag{4.5}$$

Here the characteristic velocity in the definitions of Re and Pe has been chosen as the maximum vertical velocity for $z = 0$.

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